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LETTER TO THE EDITOR

Inverse transformations and the reduction of nonlinear Dirichlet problems

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Abstract. Involutory transformations are introduced which allow the solution of a class of nonlinear Dirichlet problems. The application of coupled involutory and Bäcklund transformations is also described.

The application of Bäcklund transformations to the solution of nonlinear boundary value problems is less well developed than their role in the solution of privileged initial value problems via the inverse scattering transform. In a recent paper (Rogers 1983) reciprocal-type Bäcklund transformations were employed to reduce a class of nonlinear boundary value problems to linear canonical form. Such transformations linked to other Bäcklund transformations have been used to solve nonlinear boundary value problems in two-phase flow by Fokas and Yortsos (1982) and Rogers *et al* (1983). However, the class of boundary value problems amenable to solution by such reciprocal Bäcklund transformations is restricted to those wherein the flux is prescribed on the boundaries. Here, by contrast, involutory transformations are introduced which allow the reduction to linear canonical form of a class of nonlinear boundary value problems of Dirichlet type.

We consider the class of equations

$$\Phi(u, u_x, u_{xx}, \ldots ; u_t, u_{tt}, \ldots) := \Phi(\partial/\partial x ; \partial/\partial t ; u) = 0 \tag{1}$$

where

$$u = u(x, t). \tag{2}$$

If, as in the inverse or isotherm migration method (Ames 1972, Crank 1975) it is assumed that the relation (2) inverts to yield

$$x = x(u, t), \tag{3}$$

then we introduce the inverse transformation

$$\left.\begin{aligned} &u' = x, \\ &x' = u, \; t' = t \end{aligned}\right\} T. \tag{4}$$

Since

$$u'' = x' = u, \qquad x'' = u' = x, \qquad t'' = t' = t \tag{5}$$

it is seen that $T^2 = I$ so that the transformation is involutory.

Now,

$$\left.\frac{\partial}{\partial x}\right|_t = \left.\frac{\partial u}{\partial x}\right|_t \left.\frac{\partial}{\partial u}\right|_t + \left.\frac{\partial t}{\partial x}\right|_t \left.\frac{\partial}{\partial t}\right|_u = \left.\frac{\partial u}{\partial x}\right|_t \left.\frac{\partial}{\partial u}\right|_t \tag{6}$$

$$\left.\frac{\partial}{\partial t}\right|_x = \left.\frac{\partial u}{\partial t}\right|_x \left.\frac{\partial}{\partial u}\right|_t + \left.\frac{\partial t}{\partial t}\right|_x \left.\frac{\partial}{\partial t}\right|_u = \left.\frac{\partial u}{\partial t}\right|_x \left.\frac{\partial}{\partial u}\right|_t + \left.\frac{\partial}{\partial t}\right|_u. \tag{7}$$

In particular, (6) shows that

$$1 = \left.\frac{\partial u}{\partial x}\right|_t \left.\frac{\partial x}{\partial u}\right|_t \tag{8}$$

while (7) yields

$$\left.\frac{\partial x}{\partial t}\right|_x = 0 = \left.\frac{\partial u}{\partial t}\right|_x \left.\frac{\partial x}{\partial u}\right|_t + \left.\frac{\partial x}{\partial t}\right|_u \tag{9}$$

whence, in turn,

$$\left.\frac{\partial u}{\partial x}\right|_t = 1 \bigg/ \left.\frac{\partial x}{\partial u}\right|_t = 1 \bigg/ \left.\frac{\partial u'}{\partial x'}\right|_{t'}, \tag{10}$$

$$\left.\frac{\partial u}{\partial t}\right|_x = -\left.\frac{\partial x}{\partial t}\right|_u \bigg/ \left.\frac{\partial x}{\partial u}\right|_t = -\left.\frac{\partial u'}{\partial t'}\right|_{x'} \bigg/ \left.\frac{\partial u'}{\partial x'}\right|_{t'}. \tag{11}$$

Hence, if we introduce the operators $\mathbb{D}'$ and ∂' according to

$$\left.\frac{\partial}{\partial x}\right|_t = \left(1 \bigg/ \left.\frac{\partial u'}{\partial x'}\right|_{t'}\right) \left.\frac{\partial}{\partial x'}\right|_{t'} := \mathbb{D}' \tag{12}$$

$$\left.\frac{\partial}{\partial t}\right|_x = \left(-\left.\frac{\partial u'}{\partial t'}\right|_{x'} \bigg/ \left.\frac{\partial u'}{\partial x'}\right|_{t'}\right) \left.\frac{\partial}{\partial x'}\right|_{t'} + \left.\frac{\partial}{\partial t'}\right|_{x'} := \partial' \tag{13}$$

then, under the inverse transformation T, (1) becomes

$$\Phi(\mathbb{D}'; \partial'; x') = 0. \tag{14}$$

The above result may be used to reduce a class of nonlinear equations subject to Dirichlet boundary conditions to linear canonical form. Thus, consider the class of boundary value problems

$$\begin{aligned} &\frac{\partial u'}{\partial t'} - \gamma(u') \frac{\partial}{\partial x'}\left(\sum_{i=1}^{N} \alpha_i(u', t')\mathbb{D}'^i x'\right) = 0 \\ &u' = \Psi_1(t') \quad \text{on} \quad x' = \Phi_1(t') \\ &u' = \Psi_2(t') \quad \text{on} \quad x' = \Phi_2(t') \\ &u' = \Theta(x') \quad \text{at} \quad t' = 0. \end{aligned} \tag{15}$$

Under the inverse transformation T, the nonlinear problem (15) reduces to the dual *linear* boundary value problem

$$\begin{aligned} &\frac{\partial u}{\partial t}+\gamma(x)\frac{\partial}{\partial x}\left(\sum_{i=1}^{N}\alpha_i(x,t)\frac{\partial^i u}{\partial x^i}\right)=0 \\ &u=\Phi_1(t) \quad \text{on} \quad x=\Psi_1(t) \\ &u=\Phi_2(t) \quad \text{on} \quad x=\Psi_2(t) \\ &u=\Theta^{-1}(x) \ \text{at} \quad t=0. \end{aligned} \tag{16}$$

This reduction may be used to solve certain nonlinear moving boundary problems of the kind (15) by appeal to standard linear results. This is illustrated below.

Consider the nonlinear, moving boundary value problem

$$\begin{aligned} &\frac{\partial u'}{\partial t'}=\frac{\kappa}{(\partial u'/\partial x')^2}\frac{\partial^2 u'}{\partial x'^2} \\ &u'=0 \qquad \text{on} \quad x'=\Phi_1(t') \\ &u'=L \qquad \text{on} \quad x'=\Phi_2(t') \\ &u'=\Theta(x') \ \ \text{at} \quad t'=0. \end{aligned} \tag{17}$$

Under the inverse transformation T this reduces to the linear *fixed* boundary value problem

$$\begin{aligned} &\partial u/\partial t=\kappa\,\partial^2 u/\partial x^2 \\ &u=\Phi_1(t) \quad \text{on} \quad x=0 \\ &u=\Phi_2(t) \quad \text{on} \quad x=L \\ &u=\Theta^{-1}(x) \quad \text{at} \quad t=0 \end{aligned} \tag{18}$$

with solution

$$\begin{aligned} u=\frac{2}{L}\sum_{n=1}^{\infty}\exp(-\kappa n^2\pi^2 t/L^2)\sin\left(\frac{n\pi x}{L}\right)\Bigg[\int_0^L\Theta^{-1}(\sigma)\sin\left(\frac{n\pi\sigma}{L}\right)\mathrm{d}\sigma \\ +\frac{n\kappa\pi}{L}\int_0^t\exp(\kappa n^2\pi^2\tau/L^2)[\Phi_1(\tau)-(-1)^n\Phi_2(\tau)]\,\mathrm{d}\tau\Bigg]. \end{aligned} \tag{19}$$

Accordingly, the solution of the nonlinear problem (17) is given implicitly by

$$\begin{aligned} x'=\frac{2}{L}\sum_{n=1}^{\infty}\exp(-\kappa n^2\pi^2 t'/L^2)\sin\left(\frac{n\pi u'}{L}\right)\Bigg[\int_0^L\Theta^{-1}(\sigma)\sin\left(\frac{n\pi\sigma}{L}\right)\mathrm{d}\sigma \\ +\frac{n\kappa\pi}{L}\int_0^{t'}\exp(\kappa n^2\pi^2\tau/L^2)[\Phi_1(\tau)-(-1)^n\Phi_2(\tau)]\,\mathrm{d}\tau\Bigg]. \end{aligned} \tag{20}$$

In conclusion, it is noted that the involutory transformation T may be coupled with a Bäcklund transformation for a Burgers' hierarchy to reduce further nonlinear boundary value problems to linear canonical form. In this connection, consider the class of nonlinear problems

$$\begin{aligned}
&\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(\sum_{i=1}^{N} \alpha_i(u, t)\Psi_i\right)=0 \\
&u=\Theta_1(t) \quad \text{on} \quad x=\Phi_1(t) \\
&u=\Theta_2(t) \quad \text{on} \quad x=\Phi_2(t) \\
&u=\Theta(x) \quad \text{at} \quad t=0
\end{aligned} \tag{21}$$

where the Ψ_i are given recursively by

$$\begin{aligned}
&\Psi_i=x\Psi_{i-1}+\mathbb{D}\Psi_{i-1}, \qquad i=1,2,\ldots,N \\
&\Psi_0=1
\end{aligned} \tag{22}$$

and

$$\mathbb{D}:=\left(1\Big/\frac{\partial u}{\partial x}\right)\frac{\partial}{\partial x}.$$

Under the involutory transformation T, the boundary value problem (21) becomes

$$\begin{aligned}
&\frac{\partial u'}{\partial t'}-\frac{\partial}{\partial x'}\left(\sum_{i=1}^{N} \alpha_i(x', t')\Psi'_i\right)=0, \\
&u'=\Phi_1(t') \quad \text{on} \quad x=\Theta_1(t') \\
&u'=\Phi_2(t') \quad \text{on} \quad x'=\Theta_2(t') \\
&u'=\theta^{-1}(x') \quad \text{at} \quad t'=0
\end{aligned} \tag{23}$$

where

$$\begin{aligned}
&\Psi'_i=u'\Psi'_{i-1}+\partial\Psi'_{i-1}/\partial x' \\
&\Psi'_0=1.
\end{aligned} \tag{24}$$

The recurrence relations (24) define a Burgers' hierarchy.

Application of the Bäcklund transformation

$$\begin{aligned}
&u^*_{x^*}=u'u^* \\
&u^*_{t^*}=\left(\sum_{i=1}^{N} \alpha_i(x', t')\Psi'_i\right)u^* \\
&x^*=x', \qquad t^*=t'
\end{aligned} \tag{25}$$

takes (23) to a *linear* boundary value problem with Robin-type conditions at the moving boundaries, namely

$$\begin{aligned}
&\frac{\partial u^*}{\partial t^*}=\sum_{i=1}^{N} \alpha_i(x^*, t^*)\frac{\partial^i u^*}{\partial x^{*i}} \\
&\partial u^*/\partial x^*=\Phi_1(t^*)u^* \quad \text{on} \quad x^*=\Theta_1(t^*) \\
&\partial u^*/\partial x^*=\Phi_2(t^*)u^* \quad \text{on} \quad x^*=\Theta_2(t^*) \\
&u^*=\exp\left(\int_{x_0}^{x}\Theta^{-1}(\tau)\,\mathrm{d}\tau\right) \quad \text{at} \quad t^*=0.
\end{aligned} \tag{26}$$

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